

Splitting Monoidal Stable Model Categories

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Abstract

If \mathcal{C} is a stable model category with a monoidal product then the set of homotopy classes of self-maps of the unit forms a commutative ring, $[S, S]^{\mathcal{C}}$. An idempotent e of this ring will split the homotopy category: $[X, Y]^{\mathcal{C}} \cong e[X, Y]^{\mathcal{C}} \oplus (1-e)[X, Y]^{\mathcal{C}}$. We prove that provided the localised model structures exist, this splitting of the homotopy category comes from a splitting of the model category, that is, \mathcal{C} is Quillen equivalent to $L_{eS}\mathcal{C} \times L_{(1-e)S}\mathcal{C}$ and $[X, Y]^{L_{eS}\mathcal{C}} \cong e[X, Y]^{\mathcal{C}}$. This Quillen equivalence is strong monoidal and is symmetric when the monoidal product of \mathcal{C} is.

1 Introduction

Let R be a commutative ring with an idempotent e , so $e \cdot e = e$, then there is an equivalence of categories $R\text{-mod} \xrightarrow{\sim} eR\text{-mod} \times (1-e)R\text{-mod}$ and for any R -module M a natural isomorphism $M \cong eM \oplus (1-e)M$. This result can be useful since in general it is easier to study the categories $eR\text{-mod}$ and $(1-e)R\text{-mod}$ separately. We want to find some generalisation of this result to model categories. Our initial example is an additive and monoidal category, so we look for a class of monoidal model categories whose homotopy category is additive. The collection of monoidal stable model categories is such a class.

A pointed model category \mathcal{C} comes with a natural adjunction (Σ, Ω) on $\text{Ho}\mathcal{C}$. When this adjunction is an equivalence we say that \mathcal{C} is stable. The homotopy category of a stable model category is naturally a triangulated category (hence additive), see [Hov99, Chapter 7]. We are interested in monoidal stable model categories: those stable model categories which are also monoidal model categories ([Hov99, Section 6.6]). Thus \mathcal{C} has a closed monoidal product (\wedge, Hom) with unit S which is compatible with the model structure in the sense that the pushout product axiom holds. We write $[X, Y]^{\mathcal{C}}$ for the set of maps in the homotopy category of \mathcal{C} , this is a group since X is equivalent to $\Omega^2 \Sigma^2 X$. It is then a simple task to prove that $[S, S]^{\mathcal{C}}$ is a commutative ring (Lemma 2.1).

For any $X, Y \in \mathcal{C}$, $[X, Y]^{\mathcal{C}}$ is a $[S, S]^{\mathcal{C}}$ -module via the smash product. Hence, for any idempotent $e \in [S, S]^{\mathcal{C}}$, we have an isomorphism which is natural in X and Y : $[X, Y]^{\mathcal{C}} \cong e[X, Y]^{\mathcal{C}} \oplus (1-e)[X, Y]^{\mathcal{C}}$. Define $e\text{Ho}\mathcal{C}$ to be that category with the same

class of objects as $\mathrm{Ho}\mathcal{C}$ and with morphisms given by $e[X, Y]^{\mathcal{C}}$. Then, as with the case of R -modules above, we have an equivalence of categories $\mathrm{Ho}\mathcal{C} \xrightarrow{\sim} e\mathrm{Ho}\mathcal{C} \times (1-e)\mathrm{Ho}\mathcal{C}$.

We want to understand this splitting in terms of the model category \mathcal{C} . We assume that for any cofibrant object $E \in \mathcal{C}$ there is a new model structure on the category \mathcal{C} , written $L_E\mathcal{C}$, with the same cofibrations as \mathcal{C} and weak equivalences those maps f such that $\mathrm{Id}_E \wedge f$ is a weak equivalence of \mathcal{C} . The model structure $L_E\mathcal{C}$ is called the Bousfield localisation of \mathcal{C} at E and there is a left Quillen functor $\mathrm{Id} : \mathcal{C} \rightarrow L_E\mathcal{C}$.

For e an idempotent of $[S, S]^{\mathcal{C}}$, we are interested in localising at the objects eS and $(1-e)S$. These are constructed in terms of homotopy colimits and S is weakly equivalent to $eS \amalg (1-e)S$. Our main result, Theorem 4.4, is that the adjunction

$$\Delta : \mathcal{C} \xrightarrow{\sim} L_{eS}\mathcal{C} \times L_{(1-e)S}\mathcal{C} : \prod$$

is a Quillen equivalence. Furthermore $[X, Y]^{L_{eS}\mathcal{C}} \cong e[X, Y]^{\mathcal{C}}$, so that this Quillen equivalence induces the splitting of $\mathrm{Ho}\mathcal{C}$.

Note that there is a non-trivial idempotent $e \in [S, S]^{\mathcal{C}}$ if and only if there is a non-trivial splitting of the homotopy category. The splitting theorem proves that if there is such an idempotent, then there is a splitting of model categories. Corollary 4.5 demonstrates that if one has a splitting at the model category level (into $L_E\mathcal{C}$ and $L_F\mathcal{C}$) then the idempotent this defines (e) returns the splitting at the model category level: $L_{eS}\mathcal{C} = L_E\mathcal{C}$ and $L_{(1-e)S}\mathcal{C} = L_F\mathcal{C}$. Hence, the notions: a splitting of $[S, S]^{\mathcal{C}}$, a splitting of $\mathrm{Ho}\mathcal{C}$ and a splitting of the model category \mathcal{C} , are all equivalent.

Our motivation for this splitting result came from studying rational equivariant spectra for compact Lie groups G . The ring of self-maps of the unit in the homotopy category of rational G -spectra, $[S, S]_{\mathbb{Q}}^G$, is naturally isomorphic to the rational Burnside ring. We have a good understanding of idempotents in this ring via tom-Dieck's isomorphism, see Lemma 6.1. If a non-trivial idempotent exists, then we can use it to split the category and obtain two pieces which are possibly easier to study. We construct a model category of rational equivariant spectra in Section 5, we then give two examples of this splitting result taken from [Bar08]. Corollary 6.4 considers the case of a finite group and at the homotopy level recovers the splitting result of [GM95, Appendix A]. The second example is Lemma 6.6 and in the case of $O(2)$ the idempotent constructed is non-trivial and gives the homotopy level splitting of [Gre98].

Since we are working in a monoidal context and the splitting result is a strong monoidal adjunction, we can give two further examples: the case of modules over a ring spectrum (Proposition 7.2) and R - R -bimodules for a ring spectrum R (Proposition 7.1). After these examples we return to our motivating case of rational G -spectra and give a model structure for rational G -spectra in terms of modules over a commutative ring spectrum.

We also feel that we should mention [SS03]. In this paper the authors assume that one has a stable model category with a set of compact generators and conclude that such a category is Quillen equivalent to the category of right modules over a ring spectrum with many objects (that is, right modules over a category enriched over symmetric spectra). Consider a symmetric monoidal category \mathcal{C} with a set of compact generators \mathcal{G} such that there is an idempotent $e \in [S, S]^{\mathcal{C}}$, we can relate our splitting result to

the work of the above-mentioned paper as follows. We have two new sets of compact objects $e\mathcal{G} = \{eG \mid G \in \mathcal{G}\}$ and $(1 - e)\mathcal{G}$, their union is a set of generators for \mathcal{C} . We can construct a ring spectrum with many objects from $e\mathcal{G}$, call this $\mathcal{E}(e\mathcal{G})$. The homotopy category of right modules over $\mathcal{E}(e\mathcal{G})$ is equivalent to $e\mathrm{Ho}\mathcal{C}$ and similarly the homotopy category of right modules over $\mathcal{E}((1 - e)\mathcal{G})$ is equivalent to $(1 - e)\mathrm{Ho}\mathcal{C}$. All of our examples (see Sections 5 - 7) have a set of compact generators.

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2 Stable Model Categories

We introduce the notion of a stable model category, prove that if \mathcal{C} is a monoidal stable model category then $[S, S]^\mathcal{C}$ is a commutative ring and prove some basic results about idempotents of $[S, S]^\mathcal{C}$.

A pointed model category \mathcal{C} comes with a natural action of HosSet_* (the homotopy category of pointed simplicial sets) on $\mathrm{Ho}\mathcal{C}$, see [Hov99, Chapter 6] or [Qui67, Section I.2]. In particular for $X \in \mathcal{C}$ we have $\Sigma X := S^1 \wedge^L X$ and $\Omega X := R\mathrm{Hom}_*(S^1, X)$, these define the suspension and loop adjunction (Σ, Ω) on $\mathrm{Ho}\mathcal{C}$. When this adjunction is an equivalence we say that \mathcal{C} is **stable**, see [Hov99, Chapter 7]. Following that chapter we see that $\mathrm{Ho}\mathcal{C}$ is a triangulated category in the classical sense (see [Del77]) and that cofibre and fibre sequences agree (up to signs).

Let \mathcal{C} be a monoidal stable model category, we let \widehat{c} and \widehat{f} denote cofibrant and fibrant replacement in \mathcal{C} . For any collection of objects $\{Y_i\}_{i \in I}$ in \mathcal{C} , there is a natural map $\coprod Y_i \rightarrow \prod Y_i$. In a triangulated category finite coproducts and finite products coincide, thus when I is a finite set we have a weak equivalence $\coprod_{i \in I} \widehat{c}Y_i \rightarrow \prod_{i \in I} \widehat{f}Y_i$.

Lemma 2.1 *The set $[S, S]^\mathcal{C}$ is a commutative ring.*

Proof The homotopy category of a stable model category is additive [Hov99, Lemma 7.1.2]. Thus $[S, S]^\mathcal{C}$ is an abelian group and this addition is compatible with composition of maps $\circ : [S, S]^\mathcal{C} \otimes_\mathbb{Z} [S, S]^\mathcal{C} \rightarrow [S, S]^\mathcal{C}$. There is also a smash product operation $\wedge : [S, S]^\mathcal{C} \otimes_\mathbb{Z} [S, S]^\mathcal{C} \rightarrow [S, S]^\mathcal{C}$. The operations \circ and \wedge satisfy the following interchange law. Let a, b, c and d be elements of $[S, S]^\mathcal{C}$, then $(a \circ b) \wedge (c \circ d) = (a \wedge c) \circ (b \wedge d)$ as elements of $[S \wedge S, S \wedge S]^\mathcal{C}$ and the unit of each operation is the identity map of S . Hence, by the well-known argument below, the two operations \circ and \wedge are equal and commutative. So composition defines a commutative ring structure on the group $[S, S]^\mathcal{C}$.

Consider any set A , with two binary operations \wedge, \circ which satisfies the above interchange law. Assume there is an element $e \in A$ which acts as a both a left and right

identity for \wedge and \circ . Then $a \wedge d = (a \circ e) \wedge (e \circ d) = a \circ d$ and $a \wedge d = (e \circ a) \wedge (d \circ e) = d \circ a$. Hence the two operations are equal and are commutative. \blacksquare

Note that the above does not assume that \wedge is a *symmetric* monoidal product. Consider a map in the homotopy category, $a \in [S, S]^{\mathcal{C}}$. This can be represented by $a' : \widehat{c}fS \rightarrow \widehat{c}fS$. We can consider the homotopy colimit of the diagram $\widehat{c}fS \xrightarrow{a'} \widehat{c}fS \xrightarrow{a'} \widehat{c}fS \xrightarrow{a'} \dots$ which we denote by aS . A different choice of representative will give a weakly equivalent homotopy colimit, so we must use a little care when writing aS . The construction of the homotopy colimit aS comes with a map $\widehat{c}S \rightarrow \widehat{c}fS \rightarrow a'\widehat{c}fS$. For any $X \in \mathcal{C}$, we have the map $a' \wedge \text{Id}_X : \widehat{c}fS \wedge X \rightarrow \widehat{c}fS \wedge X$. We can then construct homotopy colimits as above to create the object aX . We use [Hov99, Proposition 7.3.2], to obtain an exact sequence:

$$0 \rightarrow \lim^1[X, Y]^{\mathcal{C}} \rightarrow [aS, Y]^{\mathcal{C}} \rightarrow \lim[X, Y]^{\mathcal{C}} \rightarrow 0.$$

We are interested in eS for e an idempotent of $[S, S]^{\mathcal{C}}$. In such a case, the \lim^1 -term is zero as the tower created by an idempotent satisfies the Mittag-Leffler condition ([Wei94, Definition 3.5.6]). Hence the above exact sequence reduces to an isomorphism $[eX, Y]^{\mathcal{C}} \rightarrow \lim[X, Y]^{\mathcal{C}} = e[X, Y]^{\mathcal{C}}$.

If e is an idempotent so is $(\text{Id}_S - e)$, which we now write as $(1 - e)$. Furthermore we have a canonical natural isomorphism $[X, Y]^{\mathcal{C}} \cong e[X, Y]^{\mathcal{C}} \oplus (1 - e)[X, Y]^{\mathcal{C}}$ for any X and Y . Thus, there is a natural isomorphism in the homotopy category $X \rightarrow eX \amalg (1 - e)X$. We can write $\text{Ho}\mathcal{C}$ as the product category $e\text{Ho}\mathcal{C} \times (1 - e)\text{Ho}\mathcal{C}$, where $e\text{Ho}\mathcal{C}$ has the same objects as $\text{Ho}\mathcal{C}$ and $e\text{Ho}\mathcal{C}(X, Y) := e[X, Y]^{\mathcal{C}}$. We wish to pull this splitting back to the level of model categories.

Lemma 2.2 *For any object X in \mathcal{C} there is a natural weak equivalence $\widehat{c}X \wedge \widehat{c}S \rightarrow \widehat{f}eX \amalg \widehat{f}(1 - e)X$.*

Proof We start with the maps $\widehat{c}X \wedge \widehat{c}S \rightarrow eX$ and $\widehat{c}X \wedge \widehat{c}S \rightarrow (1 - e)X$. By taking fibrant replacements we obtain a map $\widehat{c}X \wedge \widehat{c}S \rightarrow \widehat{f}eX \amalg \widehat{f}(1 - e)X$. The following diagram commutes for any $Y \in \mathcal{C}$, proving the result.

$$\begin{array}{ccc} [\widehat{f}eX \amalg \widehat{f}(1 - e)X, Y]^{\mathcal{C}} & \xrightarrow{\quad} & [X, Y]^{\mathcal{C}} \\ \downarrow \cong & & \downarrow \cong \\ [\widehat{f}eX \vee \widehat{f}(1 - e)X, Y]^{\mathcal{C}} & & \\ \downarrow \cong & & \\ [\widehat{f}eX, Y]^{\mathcal{C}} \oplus [\widehat{f}(1 - e)X, Y]^{\mathcal{C}} & \xrightarrow{\cong} & e[X, Y]^{\mathcal{C}} \oplus (1 - e)[X, Y]^{\mathcal{C}} \end{array}$$

\blacksquare

3 Localisations

We define the notion of a Bousfield localisation of a monoidal model category and prove that when the localisation exists, the new model category shares many of the properties of the original (left properness, the pushout product axiom and the monoid axiom). We also consider Quillen pairs between localised categories.

Recall the following concepts of localisation.

Definition 3.1 *Let E be a cofibrant object of the monoidal model category \mathcal{C} and let X, Y and Z be objects of \mathcal{C} .*

1. *A map $f : X \rightarrow Y$ is an **E -equivalence** if $\text{Id}_E \wedge f : E \wedge X \rightarrow E \wedge Y$ is a weak equivalence.*
2. *Z is **E -local** if $f^* : [Y, Z]^{\mathcal{C}} \rightarrow [X, Z]^{\mathcal{C}}$ is an isomorphism for all E -equivalences $f : X \rightarrow Y$.*
3. *An **E -localisation** of X is an E -equivalence $\lambda : X \rightarrow Y$ from X to an E -local object Y .*
4. *A is **E -acyclic** if the map $* \rightarrow A$ is an E -equivalence.*

The following is a standard result, see [Hir03, Theorems 3.2.13 and 3.2.14].

Lemma 3.2 *An E -equivalence between E -local objects is a weak equivalence.*

Consider the category \mathcal{C} with a new set of weak equivalences: the E -equivalences, while leaving the cofibrations unchanged. If this defines a model structure we call this the Bousfield localisation of \mathcal{C} at E and write it as $L_E\mathcal{C}$. The identity functor gives a strong monoidal Quillen pair (see definition below)

$$\text{Id}_{\mathcal{C}} : \mathcal{C} \rightleftarrows L_E\mathcal{C} : \text{Id}_{\mathcal{C}}.$$

This follows since the cofibrations are unchanged and if $f : X \rightarrow Y$ is an acyclic cofibration of \mathcal{C} then $f \wedge \text{Id}_E$ is also an acyclic cofibration. Hence f is a cofibration and an E -equivalence. We will write \hat{f}_E for fibrant replacement in $L_E\mathcal{C}$.

Definition 3.3 *A Quillen pair $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ between monoidal model categories is said to be a **strong monoidal adjunction** if there is a natural isomorphism $L(X \otimes Y) \rightarrow LX \otimes LY$ and an isomorphism $LS_{\mathcal{C}} \rightarrow S_{\mathcal{D}}$. We require that these isomorphisms satisfy the associativity and unital coherence conditions of [Hov99, Definition 4.1.2]. A strong monoidal adjunction (L, R) is a **strong monoidal Quillen pair** if it is a Quillen adjunction and if whenever $\hat{c}S_{\mathcal{C}} \rightarrow S_{\mathcal{C}}$ is a cofibrant replacement of $S_{\mathcal{C}}$, then the induced map $L\hat{c}S_{\mathcal{C}} \rightarrow LS_{\mathcal{C}}$ is a weak equivalence.*

From now on we assume that for any cofibrant E the E -equivalences and cofibrations define a model structure on \mathcal{C} , the E -local model structure. In general we won't have a good description of the fibrations of $L_E\mathcal{C}$, however we do have the following lemma. This result is similar in nature to [Hir03, Proposition 3.4.1].

Lemma 3.4 *An E -fibrant object is fibrant in \mathcal{C} and E -local. If X is E -local and fibrant in \mathcal{C} , then $X \rightarrow *$ has the right lifting property with respect to the class of E -acyclic cofibrations between cofibrant objects.*

Note that in many cases a stronger result holds: an object is E -fibrant if and only if it is fibrant in \mathcal{C} and E -local. For example, this stronger result holds for EKMM spectra localised at an object E by the fact that the domains of the generating E -acyclic cofibrations are cofibrant.

Proof Let $A \rightarrow B$ be an acyclic cofibration, then this is also an E -equivalence. So for an E -fibrant object Z , the canonical map $Z \rightarrow *$ will have the right lifting property with respect to $A \rightarrow B$. Let $f: A \rightarrow B$ be an E -equivalence. We must prove that $f^*: [B, Z]^{\mathcal{C}} \rightarrow [A, Z]^{\mathcal{C}}$ is an isomorphism. But since Z is E -fibrant the Quillen pair between \mathcal{C} and $L_E \mathcal{C}$ gives an isomorphism $[B, Z]^{\mathcal{C}} \cong [B, Z]^{L_E \mathcal{C}}$. This is natural in the first variable and the first statement follows.

Let $i: A \rightarrow B$ be an E -acyclic cofibration between cofibrant objects and let $f: A \rightarrow X$ be any map of \mathcal{C} . Since X is E -local, i induces an isomorphism $i^*: [B, X]^{\mathcal{C}} \rightarrow [A, X]^{\mathcal{C}}$. Choose $g: B \rightarrow X$ such that $g \circ i$ and f are homotopic. We now apply the homotopy extension property (see [Qui67, Page 1.7]), choose a path object X' for X with a map $h: A \rightarrow X'$ such that $p_0 \circ h = g \circ i$ and $p_1 \circ h = f$. We thus have the following diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X' \\ i \downarrow & & \downarrow p_0 \\ B & \xrightarrow{g} & X \end{array}$$

where i is a cofibration and p_0 is a fibration and a weak equivalence in \mathcal{C} . Thus we have a lifting $H: B \rightarrow X'$ and the map $p_1 \circ H$ is the solution to our original lifting problem. ■

If the E -local model structure exists, then every weak equivalence is an E -equivalence. Take a weak equivalence f , factor this into $g \circ h$ with h a cofibration and a weak equivalence and g an acyclic E -fibration. Then since smashing with E is a left Quillen functor, $\text{Id}_E \wedge h$ is an acyclic cofibration. By definition, $\text{Id}_E \wedge g$ is a weak equivalence, hence so is $\text{Id}_E \wedge f$. We also note that if F and E are cofibrant objects of \mathcal{C} then the model categories $L_{F \wedge E} \mathcal{C}$ and $L_E L_F \mathcal{C}$ are equal (they have the same weak equivalences and cofibrations).

Now we prove a straightforward result about Quillen functors between localised categories and then turn to proving that $L_E \mathcal{C}$ inherits many of the properties of the original model structure on \mathcal{C} .

Theorem 3.5 *Take a Quillen adjunction between monoidal model categories with a strong monoidal left adjoint $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$. Let E be cofibrant in \mathcal{C} and assume that all model categories mentioned below exist. Then (F, G) passes to a Quillen pair $F: L_E \mathcal{C} \rightleftarrows L_{FE} \mathcal{D}: G$. Furthermore, if (F, G) form a Quillen equivalence, then they pass to a Quillen equivalence of the localised categories.*

Proof Since the cofibrations in $L_E\mathcal{C}$ and $L_{FE}\mathcal{D}$ are unchanged F preserves cofibrations. Now take an acyclic cofibration in \mathcal{C} of the form $\text{Id}_E \wedge f : E \wedge X \rightarrow E \wedge Y$, applying F and using the strong monoidal condition we have a weak equivalence in \mathcal{D} : $\text{Id}_{FE} \wedge Ff : FE \wedge FX \rightarrow FE \wedge FY$. Hence F takes E -acyclic cofibrations to FE -acyclic cofibrations and we have a Quillen pair.

To prove the second statement we show that F reflects E -equivalences between cofibrant objects and that $F\hat{c}GX \rightarrow X$ is an E -equivalence for all X fibrant in $L_{FE}\mathcal{D}$. These conditions are an equivalent definition of Quillen equivalence by [Hov99, Corollary 1.3.16(b)]. The first condition follows since strong monoidality allows us to identify $F(\text{Id}_E \wedge f)$ and $\text{Id}_{FE} \wedge Ff$ for a map f in \mathcal{C} and F reflects weak equivalences between cofibrant objects. The second condition is equally simple: we know that an E -fibrant object is fibrant and that cofibrant replacement is unaffected by Bousfield localisation. Hence $F\hat{c}GX \rightarrow X$ is a weak equivalence and thus an E -equivalence. ■

Proposition 3.6 *If \mathcal{C} is left proper so is $L_E\mathcal{C}$.*

Proposition 3.7 *If \mathcal{C} is symmetric monoidal, then for two cofibrations, $f : U \rightarrow V$ and $g : W \rightarrow X$, the induced map*

$$f \square g : V \wedge W \bigvee_{U \wedge W} U \wedge X \rightarrow V \wedge X$$

is a cofibration which is an E -acyclic cofibration if either f or g is. If X is a cofibrant object then the map $\hat{c}S \wedge X \rightarrow X$ is a weak equivalence.

Proof Since the cofibrations are unchanged by localisation, we only need to check that the above map is an E -equivalence when one of f or g is. Assume that f is an E -equivalence, then the map $\text{Id}_E \wedge f : E \wedge U \rightarrow E \wedge V$ is a weak equivalence and a cofibration. Thus, since $E \wedge (-)$ commutes with pushouts the map

$$E \wedge (V \wedge W \bigvee_{U \wedge W} U \wedge X) \rightarrow E \wedge (V \wedge X)$$

is also a weak equivalence and a cofibration. By symmetry, this also deals with the case when g is an E -equivalence. The unit condition is unaffected by localisation, so it holds in the E -local model structure. ■

Thus, when \mathcal{C} is symmetric, $L_E\mathcal{C}$ is a monoidal model category. Now we consider the monoid axiom.

Proposition 3.8 *If \mathcal{C} is symmetric monoidal and satisfies the monoid axiom, then so does $L_E\mathcal{C}$.*

Proof Let $i : A \rightarrow X$ be an acyclic E -cofibration, then for any object Y , the map $\text{Id}_E \wedge i \wedge \text{Id}_Y$ is a weak equivalence. Moreover, transfinite compositions of pushouts of such maps are weak equivalences by the monoid axiom for \mathcal{C} . Thus transfinite compositions of pushouts of maps of the form $i \wedge \text{Id}_Y$ are E -equivalences. ■

4 The Splitting

We are now ready to prove our main result, Theorem 4.4. We conclude this section with a converse to this result.

Recall the definition of the product model category from [Hov99, Example 1.1.6]. Given model categories M_1 and M_2 we can put a model category structure on $M_1 \times M_2$. A map (f_1, f_2) is a cofibration, weak equivalence or fibration if and only if f_1 is so in M_1 and f_2 is so in M_2 . Similarly a finite product of model categories has a model structure where a map is a cofibration, weak equivalence or fibration if and only if each of its factors is so. If M_1 and M_2 both satisfy any of the following: left properness, right properness, the pushout product axiom, the monoid axiom or cofibrant generation, then so does $M_1 \times M_2$.

Proposition 4.1 *If E and F are cofibrant objects of \mathcal{C} then there is a strong monoidal Quillen adjunction*

$$\Delta : \mathcal{C} \rightleftarrows L_E \mathcal{C} \times L_F \mathcal{C} : \prod.$$

Let \mathcal{C} be a stable monoidal model category with an idempotent $e \in [S, S]^{\mathcal{C}}$. Then we have a Quillen pair

$$\Delta : \mathcal{C} \rightleftarrows L_{eS} \mathcal{C} \times L_{(1-e)S} \mathcal{C} : \prod$$

and an equivalence of homotopy categories

$$\Delta : \mathrm{Ho} \mathcal{C} \rightleftarrows_e \mathrm{Ho} \mathcal{C} \times (1-e) \mathrm{Ho} \mathcal{C} : \prod.$$

We now wish to prove that the Quillen pair induces this equivalence of homotopy categories.

Lemma 4.2 *Take an idempotent $e \in [S, S]^{\mathcal{C}}$, any pair of objects X, Y and an eS -local object Z . Then there are natural isomorphisms*

$$[X, Y]^{L_{eS} \mathcal{C}} \longrightarrow [X, \widehat{f}_{eS} Y]^{\mathcal{C}}, \quad [X, Z]^{\mathcal{C}} \longrightarrow e[X, Z]^{\mathcal{C}}.$$

Proof The first comes from the Quillen adjunction between \mathcal{C} and $L_{eS} \mathcal{C}$. For the second we use the fact that the map $\widehat{c}X \rightarrow eX$ is an eS -equivalence to obtain isomorphisms $[X, Z]^{\mathcal{C}} \leftarrow [eX, Z]^{\mathcal{C}} \rightarrow e[X, Z]^{\mathcal{C}}$. ■

Lemma 4.3 *Let e be an idempotent of $[S, S]^{\mathcal{C}}$. Then the map $e : eS \rightarrow eS$ is an isomorphism in $\mathrm{Ho} \mathcal{C}$. Hence $(1-e) : eS \rightarrow eS$ is equal to the zero map in $\mathrm{Ho} \mathcal{C}$ and so for any X and Y in \mathcal{C} , $(1-e)[X, eY]^{\mathcal{C}} = 0$.*

Proof Consider the map $e^* : [eS, X]^{\mathcal{C}} \rightarrow [eS, X]^{\mathcal{C}}$, this is naturally isomorphic to $e^* : e[S, X]^{\mathcal{C}} \rightarrow e[S, X]^{\mathcal{C}}$, which is an isomorphism. The second part follows since $(1-e) \circ e \in [S, S]^{\mathcal{C}}$ is equal to zero. ■

Theorem 4.4 *Let \mathcal{C} be a stable monoidal model category with an idempotent $e \in [S, S]^{\mathcal{C}}$. Assume that the model categories $L_{eS} \mathcal{C}$ and $L_{(1-e)S} \mathcal{C}$ exist, then the strong monoidal Quillen pair below is a Quillen equivalence.*

$$\Delta : \mathcal{C} \rightleftarrows L_{eS} \mathcal{C} \times L_{(1-e)S} \mathcal{C} : \prod$$

Proof The right adjoint detects all weak equivalences: take $f:A \rightarrow B$ in $L_{eS}\mathcal{C}$ and $g:C \rightarrow D$ in $L_{(1-e)S}\mathcal{C}$. If $(f,g):A \amalg C \rightarrow B \amalg D$ is a weak equivalence then f and g are weak equivalences since they are retracts of (f,g) . Hence f is an eS -equivalence and g is a $(1-e)S$ -equivalence.

Let X be a cofibrant object of \mathcal{C} , we then have an eS -acyclic cofibration $X \rightarrow \widehat{f}_{eS}X$ and an $(1-e)S$ -acyclic cofibration $X \rightarrow \widehat{f}_{(1-e)S}X$. We must prove that $X \rightarrow \widehat{f}_{eS}X \amalg \widehat{f}_{(1-e)S}X$ is a weak equivalence. For any $A \in \mathcal{C}$ we have the following commutative diagram:

$$\begin{array}{ccc} e[A, X]^{\mathcal{C}} \oplus (1-e)[A, X]^{\mathcal{C}} & \longrightarrow & e[A, \widehat{f}_{eS}X]^{\mathcal{C}} \oplus (1-e)[A, \widehat{f}_{(1-e)S}X]^{\mathcal{C}} \\ \uparrow \cong & & \uparrow \cong \\ [A, X]^{\mathcal{C}} & \longrightarrow & [A, \widehat{f}_{eS}X \amalg \widehat{f}_{(1-e)S}X]^{\mathcal{C}} \end{array}$$

So we have reduced the problem to proving that $e[A, X]^{\mathcal{C}} \rightarrow e[A, \widehat{f}_{eS}X]^{\mathcal{C}}$ is an isomorphism. This follows from the commutative diagram below and Lemma 4.3, which tells us that the terms $e[A, (1-e)X]^{\mathcal{C}}$ and $e[A, (1-e)\widehat{f}_{eS}X]^{\mathcal{C}}$ are zero.

$$\begin{array}{ccc} e[A, X]^{\mathcal{C}} & \longrightarrow & e[A, \widehat{f}_{eS}X]^{\mathcal{C}} \\ \downarrow \cong & & \downarrow \cong \\ e[A, eX]^{\mathcal{C}} \oplus e[A, (1-e)X]^{\mathcal{C}} & \xrightarrow{\cong} & e[A, e\widehat{f}_{eS}X]^{\mathcal{C}} \oplus e[A, (1-e)\widehat{f}_{eS}X]^{\mathcal{C}} \end{array}$$

■

A finite orthogonal decomposition of Id_S is a collection of idempotents e_1, \dots, e_n which sum to the identity in $[S, S]^{\mathcal{C}}$ such that $e_i \circ e_j = 0$ for $i \neq j$. This result extends to give a strong monoidal Quillen equivalence between \mathcal{C} and $\prod_{i=1}^n L_{e_i S}\mathcal{C}$ whenever e_1, \dots, e_n is a finite orthogonal decomposition of Id_S .

Corollary 4.5 *Consider a monoidal model category \mathcal{C} which splits as a product $L_E\mathcal{C} \times L_F\mathcal{C}$, for cofibrant objects E and F . Then there are orthogonal idempotents e_E and e_F in $[S, S]^{\mathcal{C}}$ such that $e_E + e_F = \text{Id}_S$, $L_{e_E S}\mathcal{C} = L_E\mathcal{C}$ and $L_{e_F S}\mathcal{C} = L_F\mathcal{C}$.*

Proof Using the isomorphism $[S, S]^{L_E\mathcal{C}} \oplus [S, S]^{L_F\mathcal{C}} \rightarrow [S, S]^{\mathcal{C}}$ define e_E as the image of $\text{Id}_S \oplus 0 \in [S, S]^{L_E\mathcal{C}} \oplus [S, S]^{L_F\mathcal{C}}$ in $[S, S]^{\mathcal{C}}$. Similarly define e_F as the image of $0 \oplus \text{Id}_S$. Thus we have idempotents e_E and e_F in $[S, S]^{\mathcal{C}}$ such that $e_E + e_F = \text{Id}_S$ and $e_E \circ e_F = 0$. By construction, $e_E[X, Y]^{\mathcal{C}} \cong [X, Y]^{L_E\mathcal{C}}$ and by our work above $e_E[X, Y]^{\mathcal{C}} \cong [X, Y]^{L_{e_E S}\mathcal{C}}$. From this it follows that the $e_E S$ -equivalences are the E -equivalences and $L_{e_E S}\mathcal{C} = L_E\mathcal{C}$. ■

5 Rational Equivariant Spectra

Our motivating example for the splitting result is the category of rational G -equivariant EKMM S -modules for a compact Lie group G . Our first task is to define this category,

for this we will need a rational sphere spectrum. We work with GM , the category of G -equivariant EKMM S -modules from [MM02]. One could work with G -equivariant orthogonal spectra and perform analogous constructions there and obtain equivalent results for that category. In particular the two categories of equivariant spectra we have mentioned are monoidally Quillen equivalent.

We will construct \mathbb{Q} as a group and translate this into spectra. Take a free resolution of \mathbb{Q} as an abelian group, $0 \rightarrow R \xrightarrow{f} F \rightarrow \mathbb{Q} \rightarrow 0$, where $F = \bigoplus_{q \in \mathbb{Q}} \mathbb{Z}$. Since a free abelian group is a direct sum of copies of \mathbb{Z} we can rewrite this short exact sequence as $0 \rightarrow \bigoplus_i \mathbb{Z} \xrightarrow{f} \bigoplus_j \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow 0$. Since \mathbb{Q} is flat, the sequence $0 \rightarrow \bigoplus_i M \xrightarrow{f \otimes \text{Id}} \bigoplus_j M \rightarrow \mathbb{Q} \otimes M \rightarrow 0$ is exact for any abelian group M . Hence for each subgroup H of G , we have an injective map (which we also denote as f) $\bigoplus_i A(H) \xrightarrow{f \otimes \text{Id}} \bigoplus_j A(H)$ and $\bigoplus_j A(H) / \bigoplus_i A(H) \cong A(H) \otimes \mathbb{Q}$. For H , a subgroup of G ,

$$[\bigvee_i S, \bigvee_j S]^H \cong \text{Hom}_{A(H)} \left(\bigoplus_i A(H), \bigoplus_j A(H) \right).$$

Thus we can choose $g : \bigvee_i \widehat{c}S \rightarrow \bigvee_j \widehat{c}S$, a representative for the homotopy class corresponding to f .

Let I be the unit interval with basepoint 0, there is a cofibration of spaces $S^0 \rightarrow I$ which sends the non-basepoint point of S^0 to $1 \in I$. If X is a cofibrant G -spectrum then $X \cong X \wedge S^0 \rightarrow X \wedge I$ is a cofibration since G -spectra are enriched over spaces (see [MM02, Chapter III, Definition 1.14] and [Hov99, Lemma 4.2.2]). For a map $f : X \rightarrow Y$, the cofibre of f , C_f , is the pushout of the diagram $X \wedge I \leftarrow X \xrightarrow{f} Y$. If X is cofibrant then the map $Y \rightarrow C_f$ is a cofibration, hence if X and Y are cofibrant, so is C_f .

Definition 5.1 *For the map g as constructed above, the cofibre of g is the **rational sphere spectrum** and we have a cofibre sequence*

$$\bigvee_i \widehat{c}S \xrightarrow{g} \bigvee_j \widehat{c}S \longrightarrow S_{\mathcal{M}}^0 \mathbb{Q}.$$

A different choice of representative for the homotopy class $[g]$ will induce a weak equivalence between the cofibres, and hence (up to weak equivalence) $S_{\mathcal{M}}^0 \mathbb{Q}$ is independent of this choice of representative. Note that there is an inclusion $\alpha : \widehat{c}S \rightarrow \bigvee_j \widehat{c}S$ which sends $\widehat{c}S$ to the term of $\bigvee_j \widehat{c}S$ corresponding to $1 \in \mathbb{Q}$.

Proposition 5.2 *Let X be a G -spectrum, then for any subgroup H of G the map $(\text{Id}_X \wedge \alpha)_* : \pi_*^H(X) \rightarrow \pi_*^H(X \wedge S_{\mathcal{M}}^0 \mathbb{Q})$ induces an isomorphism $\pi_*^H(X) \otimes \mathbb{Q} \rightarrow \pi_*^H(X \wedge S_{\mathcal{M}}^0 \mathbb{Q})$.*

Proof Using the cofibre sequence which defines $S_{\mathcal{M}}^0 \mathbb{Q}$ we have the following collection

of isomorphic long exact sequences of homotopy groups

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \pi_n^H(X \wedge \bigvee_i \widehat{c}S) & \xrightarrow{(\text{Id} \wedge g)^*} & \pi_n^H(X \wedge \bigvee_j \widehat{c}S) & \longrightarrow & \pi_n^H(X \wedge S_{\mathcal{M}}^0 \mathbb{Q}) & \longrightarrow \cdots \\
\cdots \longrightarrow & \pi_n^H(\bigvee_i X) & \xrightarrow{(\text{Id} \wedge g)^*} & \pi_n^H(\bigvee_j X) & \longrightarrow & \pi_n^H(X \wedge S_{\mathcal{M}}^0 \mathbb{Q}) & \longrightarrow \cdots \\
\cdots \longrightarrow & \bigoplus_i \pi_n^H(X) & \xrightarrow{g \otimes \text{Id}} & \bigoplus_j \pi_n^H(X) & \longrightarrow & \pi_n^H(X \wedge S_{\mathcal{M}}^0 \mathbb{Q}) & \longrightarrow \cdots \\
\cdots \longrightarrow & \bigoplus_i \mathbb{Z} \otimes \pi_n^H(X) & \xrightarrow{g \otimes \text{Id}} & \bigoplus_j \mathbb{Z} \otimes \pi_n^H(X) & \longrightarrow & \pi_n^H(X \wedge S_{\mathcal{M}}^0 \mathbb{Q}) & \longrightarrow \cdots
\end{array}$$

Since the map $g \otimes \text{Id} : (\bigoplus_i \mathbb{Z}) \otimes \pi_n^H(X) \rightarrow (\bigoplus_j \mathbb{Z}) \otimes \pi_n^H(X)$ is injective for all n , this long exact sequence splits into short exact sequences and the result follows. \blacksquare

There are many other methods for constructing a rational sphere spectrum, these will all be weakly equivalent to $S_{\mathcal{M}}^0 \mathbb{Q}$ as we prove below. One obvious alternative is to construct a homotopy colimit of the diagram $\widehat{c}S \xrightarrow{2} \widehat{c}S \xrightarrow{3} \widehat{c}S \xrightarrow{4} \cdots$, call this object $R_{\mathbb{Q}}$. It follows that the map $\pi_*^H(\widehat{c}X) \rightarrow \pi_*^H(R_{\mathbb{Q}} \wedge \widehat{c}X)$ induced by $\widehat{c}S \rightarrow R_{\mathbb{Q}}$ gives an isomorphism $\pi_*^H(\widehat{c}X) \otimes \mathbb{Q} \rightarrow \pi_*^H(R_{\mathbb{Q}} \wedge \widehat{c}X)$. We prove in Lemma 5.9 that if you have any rationalisation of the sphere – a rational equivalence $f : S \rightarrow X$ where X is a spectrum with $\pi_*^H(X)$ rational for all n and H , then $S_{\mathcal{M}}^0 \mathbb{Q}$ and X are weakly equivalent.

The result below is [MM02, Chapter IV, Theorem 6.3], the proof of which is an adaptation of the material in [EKMM97, chapter VIII].

Theorem 5.3 *Let E be a cofibrant spectrum or a cofibrant based G -space. Then $G\mathcal{M}$ has an E -model structure whose weak equivalences are the E -equivalences and whose E -cofibrations are the cofibrations of $G\mathcal{M}$. The E -fibrant objects are precisely the E -local objects and E -fibrant approximation constructs a Bousfield localisation $f_X : X \rightarrow \widehat{f}_E X$ of X at E . The notation for E -model structure on the underlying category of $G\mathcal{M}$ is $L_E G\mathcal{M}$ or $G\mathcal{M}_E$.*

The categories $L_E G\mathcal{M}$ are cofibrantly generated model categories, this is implied by the proof of [EKMM97, Chapter VIII, Theorem 1.1]. Let c be a fixed infinite cardinal that is at least the cardinality of $E^*(S)$. Then define \mathcal{T} , a test set for E -fibrations, to consist of all inclusions of cell complexes $X \rightarrow Y$ such that the cardinality of the set of cells of Y is less than or equal to c . Hence the domains of these maps are κ -small where κ is the least cardinal greater than c . Thus if we let I be the set of generating cofibrations for $G\mathcal{M}$, then we can take I and \mathcal{T} as sets of generating cofibrations and generating acyclic cofibrations for $L_E G\mathcal{M}$.

Lemma 5.4 *For a map $g : X \rightarrow Y$ the following are equivalent:*

1. $g : X \rightarrow Y$ is an $S_{\mathcal{M}}^0 \mathbb{Q}$ -equivalence.
2. $g_*^H : \pi_*(X^H) \otimes \mathbb{Q} \rightarrow \pi_*(Y^H) \otimes \mathbb{Q}$ is an isomorphism for all H .
3. $g_*^H : H_*(X^H; \mathbb{Q}) \rightarrow H_*(Y^H; \mathbb{Q})$ is an isomorphism for all H .

Proof We have shown in Proposition 5.2 that the first two conditions are equivalent. The last two statements are equivalent since the Hurewicz map induces an isomorphism $\pi_*(A) \otimes \mathbb{Q} \rightarrow H_*(A; \mathbb{Q})$ for any non-equivariant spectrum A . ■

Definition 5.5 *The model category of rational G -spectra is defined to be $L_{S_{\mathcal{M}}^0 \mathbb{Q}} G\mathcal{M}$, which we write as $GM_{\mathbb{Q}}$. Since the $S_{\mathcal{M}}^0 \mathbb{Q}$ -equivalences are precisely the rational homotopy isomorphisms, we call the $S_{\mathcal{M}}^0 \mathbb{Q}$ -equivalences **rational equivalences** or $\pi_*^{\mathbb{Q}}$ -**isomorphisms**. The set of rational homotopy classes of maps from X to Y will be written $[X, Y]_{\mathbb{Q}}^G$ and we will write $\widehat{f}_{\mathbb{Q}}$ for fibrant replacement in the localised category.*

The lemma above proves that our model structure is independent of our choice of rational sphere spectrum. We now prove that $GM_{\mathbb{Q}}$ is a right proper model category, for which we need the following.

Lemma 5.6 *For any map $f : X \rightarrow Y$ of G -prespectra and any $H \subset G$, there are natural long exact sequences*

$$\begin{aligned} \cdots \rightarrow \pi_q^H(Ff) \otimes \mathbb{Q} \rightarrow \pi_q^H(X) \otimes \mathbb{Q} \rightarrow \pi_q^H(Y) \otimes \mathbb{Q} \rightarrow \pi_{q-1}^H(Ff) \otimes \mathbb{Q} \rightarrow \cdots, \\ \cdots \rightarrow \pi_q^H(X) \otimes \mathbb{Q} \rightarrow \pi_q^H(Y) \otimes \mathbb{Q} \rightarrow \pi_q^H(Cf) \otimes \mathbb{Q} \rightarrow \pi_{q-1}^H(X) \otimes \mathbb{Q} \rightarrow \cdots \end{aligned}$$

and the natural map $\nu : Ff \rightarrow \Omega Cf$ is a π_* -isomorphism.

Proof By [MM02, Chapter IV, Remark 2.8], we have long exact sequences as above, but without needing to tensor with \mathbb{Q} . Since \mathbb{Q} is flat, tensoring with it preserves exact sequences, hence the result follows. ■

Lemma 5.7 *The category $GM_{\mathbb{Q}}$ is right proper.*

Proof Following the proof of [MMSS01, Lemma 9.10] one shows that a stronger statement holds: in a pullback diagram as below, if β is a level wise fibration of G -spaces then r is a $\pi_*^{\mathbb{Q}}$ -isomorphism.

$$\begin{array}{ccc} W & \xrightarrow{\delta} & X \\ r \downarrow & \lrcorner & \downarrow \sim_{\mathbb{Q}} \\ Y & \xrightarrow{\beta} & Z \end{array}$$

The only point of difference is that in the last step of the proof one needs to use the long exact sequence of *rational* homotopy groups of a fibration. ■

Since our localisation is of a particularly nice form, we are able to give the following interpretation of maps in $\mathrm{Ho} GM_{\mathbb{Q}}$.

Theorem 5.8 *For any X and Y , $[X, Y]_{\mathbb{Q}}^G$ is a rational vector space. If Z is an $S_{\mathcal{M}}^0 \mathbb{Q}$ -local object of GM then Z has rational homotopy groups. There is a natural isomorphism $[X, Y]_{\mathbb{Q}}^G \cong [X \wedge S_{\mathcal{M}}^0 \mathbb{Q}, Y \wedge S_{\mathcal{M}}^0 \mathbb{Q}]^G$.*

Proof For each integer n we have a self-map of $\widehat{c}S$ which represents multiplication by n at the model category level, applying $(-)\wedge X$ we obtain a self-map of $\widehat{c}S\wedge X$. Since this map is an isomorphism of rational homotopy groups it induces an isomorphism $n:[X,Y]_{\mathbb{Q}}^G \rightarrow [X,Y]_{\mathbb{Q}}^G$, hence $[X,Y]_{\mathbb{Q}}^G$ is a rational vector space. The homotopy groups of Z can be given in terms of $[\Sigma^p G/H_+, Z]^G$ for p an integer and H a subgroup of G . Since we have assumed that Z is $S_{\mathcal{M}}^0\mathbb{Q}$ -local, this homotopy group is isomorphic to $[\Sigma^p G/H_+, Z]_{\mathbb{Q}}^G$ which we now know is a rational vector space.

The map $Y\wedge S_{\mathcal{M}}^0\mathbb{Q} \rightarrow \widehat{f}_{\mathbb{Q}}(Y\wedge S_{\mathcal{M}}^0\mathbb{Q})$ is a $\pi_*^{\mathbb{Q}}$ -isomorphism between objects with rational homotopy groups, hence it is a π_* -isomorphism. For any G -spectrum X , $X\wedge S_{\mathcal{M}}^0\mathbb{Q}$ is rationally equivalent to X . Combining these we obtain isomorphisms as below.

$$\begin{aligned} [X,Y]_{\mathbb{Q}}^G &\cong [X\wedge S^0\mathbb{Q}, Y\wedge S^0\mathbb{Q}]_{\mathbb{Q}}^G \\ &\cong [X\wedge S^0\mathbb{Q}, \widehat{f}_{\mathbb{Q}}(Y\wedge S^0\mathbb{Q})]^G \\ &\cong [X\wedge S^0\mathbb{Q}, Y\wedge S^0\mathbb{Q}]^G \end{aligned}$$

■

The following result gives a universal property for $S_{\mathcal{M}}^0\mathbb{Q}$. Note that if the map f is a rational equivalence, then the lift in the proof below is a rational equivalence between spectra with rational homotopy groups and hence is a weak equivalence.

Lemma 5.9 *Let X be a spectrum with a map $f:S \rightarrow X$ such that $\pi_n^H(X)$ is a rational vector space for each subgroup H and integer n . Then there is a map $S_{\mathcal{M}}^0\mathbb{Q} \rightarrow X$ in $\text{Ho } G\mathcal{M}$ such that the composite $S \rightarrow S_{\mathcal{M}}^0\mathbb{Q} \rightarrow X$ is equal to the map f (in $\text{Ho } G\mathcal{M}$).*

Proof By Theorem 5.8 the map $\widehat{c}X \rightarrow \widehat{f}_{\mathbb{Q}}\widehat{c}X$ is a weak equivalence. We then draw the diagram below and obtain a lifting $S_{\mathcal{M}}^0\mathbb{Q} \rightarrow \widehat{f}_{\mathbb{Q}}\widehat{c}X$ using the rational model structure on $G\mathcal{M}$.

$$\begin{array}{ccccc} \widehat{c}S & \longrightarrow & \widehat{c}X & \xrightarrow{\sim} & \widehat{f}_{\mathbb{Q}}\widehat{c}X \\ \downarrow \sim_{\mathbb{Q}} & & & & \downarrow \\ S_{\mathcal{M}}^0\mathbb{Q} & \longrightarrow & & & * \end{array}$$

■

6 Splitting Rational Equivariant Spectra

We show how splittings of the category of rational equivariant spectra correspond to idempotents of the rational Burnside ring. In particular, we know all such idempotents in the case of a finite group and we have the idempotent e_1 , constructed in Lemma 6.6, which is in many cases a non-trivial idempotent. For a compact Lie group G the Burnside ring is defined to be $[S, S]^G$. The following result is tom Dieck's isomorphism, see [LMSM86, Chapter V, Lemma 2.10] which references [tD77, Lemma 6]. This result can be very useful when studying the Burnside ring of G . Recall that $\mathcal{F}G$ is the set of subgroups of G that have finite index in their normaliser. There is a topology on $\mathcal{F}G$ (induced by the Hausdorff metric on subsets of G) such that the conjugation action of G on $\mathcal{F}G$ is continuous, see [LMSM86, Chapter V, Lemma 2.8].

Lemma 6.1 *Let $C(\mathcal{F}G/G, \mathbb{Q})$ denote the ring of continuous maps from the orbit space $\mathcal{F}G/G$ to \mathbb{Q} , where \mathbb{Q} is considered as a topological space with the discrete topology. The map $[S, S]^G \rightarrow C(\mathcal{F}G/G, \mathbb{Q})$ which takes f to $(H) \mapsto \deg(f^H)$ induces an isomorphism of rings $[S, S]^G \otimes \mathbb{Q} \rightarrow C(\mathcal{F}G/G, \mathbb{Q})$.*

In particular, for a finite group G , this specifies an isomorphism $[S, S]^G \otimes \mathbb{Q} \rightarrow \prod_{(H) \leq G} \mathbb{Q}$. Let $e_H \in [S, S]^G \otimes \mathbb{Q}$ be the idempotent corresponding to projection onto factor (H) , then we have a finite orthogonal decomposition of Id_S given by the collection $\{e_H\}$ as H runs over the conjugacy classes of subgroups of G . We now give an isomorphism between the rational Burnside ring and self maps of S in $\text{Ho } GM_{\mathbb{Q}}$.

Proposition 6.2 *There is a ring isomorphism $[S, S]^G \otimes \mathbb{Q} \rightarrow [S, S]_{\mathbb{Q}}^G$ induced by $\text{Id} : GM \rightarrow GM_{\mathbb{Q}}$.*

Proof The identity functor induces a ring map $[S, S]^G \rightarrow [S, S]_{\mathbb{Q}}^G$ and since the right hand side is a rational vector space this induces the desired map of rings. That this map is an isomorphism follows from the isomorphisms: $[S, S]^G \otimes \mathbb{Q} \cong [S, S_{\mathcal{M}}^0 \mathbb{Q}]^G$, $[S, S_{\mathcal{M}}^0 \mathbb{Q}]^G \cong [S, \widehat{f}_{\mathbb{Q}} S]^G$ and $[S, \widehat{f}_{\mathbb{Q}} S]^G \cong [S, S]_{\mathbb{Q}}^G$. The universal property of $S_{\mathcal{M}}^0 \mathbb{Q}$ provides the second isomorphism and ensures that the composite of the above maps is equal to the specified map of rings. ■

Corollary 6.3 *If e is an idempotent of the rational Burnside ring of G , then the adjunction below is a strong symmetric monoidal Quillen equivalence.*

$$\Delta : GM_{\mathbb{Q}} \rightleftarrows L_{eS} GM_{\mathbb{Q}} \times L_{(1-e)S} GM_{\mathbb{Q}} : \prod$$

Corollary 6.4 *The category of rational G -spectra (for finite G) splits into the product of the localisations $L_{e_H S} G\mathcal{J}\mathcal{S}_{\mathbb{Q}}$ as (H) runs over the conjugacy classes of subgroups of G .*

At the homotopy level this result can be found in [GM95, Appendix A]. Note that the two localisations of G -spectra that we have used: $L_{S_{\mathcal{M}}^0 \mathbb{Q}} GM$ and $L_{eS} L_{S_{\mathcal{M}}^0 \mathbb{Q}} GM$ share many of the same properties. This is because they are designed to invert elements of $[S, S]^G$ and $[S, S]_{\mathbb{Q}}^G$ respectively. The first is designed to invert the primes and the second inverts the idempotent e .

Lemma 6.5 *For e an idempotent of $[S, S]^G \otimes \mathbb{Q}$ the category $L_{eS} GM$ is right proper.*

Proof Let $e \in [S, S]^G \otimes \mathbb{Q}$ be an idempotent, then for any exact sequence of $[S, S]^G \otimes \mathbb{Q}$ -modules $\cdots \rightarrow M_i \rightarrow M_{i-1} \rightarrow \cdots$, the sequence $\cdots \rightarrow eM_i \rightarrow eM_{i-1} \rightarrow \cdots$ is exact. Right properness then follows from the proof of Lemma 5.7 by applying e to the long exact sequence of rational homotopy groups of a fibration. ■

We now give a general example of an idempotent of the Burnside ring. This idempotent is non-trivial in many cases, such as when $G = O(2)$, the group of two-by-two orthogonal matrices. This idempotent was used to study rational $O(2)$ -spectra in [Gre98] and [Bar08, Part III].

Lemma 6.6 *Let G be a compact Lie group and let S denote the set of subgroups of the identity component of G which have finite index in their normaliser. Then there is an idempotent $e_1 \in [S, S]^G \otimes \mathbb{Q} \cong C(\mathcal{F}G/G, \mathbb{Q})$ given by the map which sends (H) to 1 if $H \in S$ and zero otherwise.*

Proof Let G_1 denote the identity component of G and recall that since G is compact $F = G/G_1$ is finite. Take $H \in S$, by [Bre72, Chapter II, Corollary 5.6] we know that if $K \in \mathcal{F}G$ is in some sufficiently small neighbourhood of H in the space $\mathcal{F}G$, then K is subconjugate to H and so K is a subgroup of G_1 . It follows that S is open in $\mathcal{F}G/G$. Now take (K) to be in $(\mathcal{F}G/G) \setminus S$, so there is a $g \in G \setminus G_1$ such that $K \cap gG_1$ is non-empty. Then any $L \in \mathcal{F}G$ that is sufficiently close to K also has a non-trivial intersection with gG_1 so L is not a subgroup of G_1 , it follows that S is also closed. Hence e_1 , the characteristic function of S , is a continuous map $\mathcal{F}G/G \rightarrow \mathbb{Q}$. Thus e_1 is an idempotent, since $e_1(H) = 1$ if $H \in S$ and zero otherwise. ■

Let \mathcal{F} be the set of subgroups of G_1 , then it can be shown that $e_1 S$ is weakly equivalent to $E\mathcal{F}_+$ (the universal space for a family). One can then use the results of [MM02, Chapter IV, Section 6] to obtain better understanding of $L_{e_1 S} GM_{\mathbb{Q}}$ and $L_{(1-e_1)S} GM_{\mathbb{Q}}$.

7 Modules and Bimodules

We give two general examples of where our splitting result can be applied. Choose a monoidal model category of spectra, such as symmetric, orthogonal or EKMM spectra (this could even be G -equivariant for the last two versions) and call it \mathcal{S} . For R a ring spectrum we consider splittings of the model category of R - R -bimodules, this is a monoidal model category which is not (in general) symmetric. We let $[-, -]^{(R, R)}$ denote maps in the homotopy category of R - R -bimodules. Our second example considers the case of R -modules, when R is not commutative. Although R -mod is not a monoidal model category we can still obtain splittings of the model category by considering idempotents of $[R, R]^{(R, R)}$. We return to rational equivariant spectra at the end of this section and create a commutative ring spectrum $S_{\mathbb{Q}}$ such that $S_{\mathbb{Q}}$ -mod is Quillen equivalent to $GM_{\mathbb{Q}}$ (Theorem 7.6). We then show that splittings of $S_{\mathbb{Q}}$ -mod correspond to splittings of $GM_{\mathbb{Q}}$.

We first introduce some results from [EKMM97], these can be adapted to any of the categories of spectra we have mentioned above. For R an algebra, there is a notion of a cell R -module, see [EKMM97, Chapter III, Definition 2.1], a cell R module is a special kind of cofibrant module. We can always replace an R -module M by a weakly equivalent cell R -module ΓM via [EKMM97, Chapter III, Theorem 2.10].

If E is a right R -module then we have a spectrum $E \wedge_R X$ for any left R -module X . It is defined as the coequaliser of the diagram $E \wedge R \wedge X \rightrightarrows E \wedge X$ where the maps are given by the action of R on E and the action of R on X . Thus we have the notion of an E^R -equivalence of R -modules: a map f in R -mod such that $E \wedge_R f$ is a weak equivalence of underlying spectra. Let E be a cell right R -module, then by [EKMM97, Chapter VIII, Theorem 1.1], there is a model structure $L_E R$ -mod on the category of

R -modules with weak equivalences the E^R -equivalences and cofibrations given by the cofibrations for $R\text{-mod}$. We also note that if X is a cofibrant R -module, the functor $-\wedge_R X$ preserves weak equivalences ([EKMM97, Chapter III, Theorem 3.8]).

Proposition 7.1 *For R a ring spectrum in \mathcal{S} , whose underlying spectrum is cofibrant, an idempotent of $THH^0(R) := [R, R]^{(R, R)}$ splits the category of R - R -bimodules.*

Proof We can identify the category of R - R bimodules with the category of $R \wedge R^{op}$ -modules. The ring spectrum R^{op} has the same underlying spectrum as R but the multiplication is given by $R \wedge R \xrightarrow{\tau} R \wedge R \xrightarrow{\mu} R$ where τ is the symmetry isomorphism of \wedge in \mathcal{S} and μ is the multiplication of R . We have assumed that R is cofibrant to ensure that $R \wedge R^{op}$ is weakly equivalent to $R \wedge^L R^{op}$, thus $[X, Y]^{(R, R)} \cong [X, Y]^{R \wedge^L R^{op}}$.

For a cell R - R -bimodule E we have a E -local model structure on the category of R - R -bimodules. If M is a cofibrant R - R -bimodule, then an M -equivalence is the same as a ΓM -equivalence and so we can localise at any cofibrant bimodule by localising at its cellular replacement. We can now apply Theorem 4.4 to complete the proof. ■

We now turn to left modules over a ring spectrum, we can obtain a splitting result when R is not commutative. In this case $R\text{-mod}$ does not have a monoidal product and so $[R, R]^R$ does not act on $[X, Y]^R$. Instead we will use the action of $[R, R]^{(R, R)}$ on $[X, Y]^R$ to split the category. Throughout we assume that R is cofibrant as a spectrum.

We return to algebra briefly to offer some context for this result. If R was an arbitrary ring, then for a *central* idempotent $e \in R$, (so $er = re$ for any $r \in R$), one can form new rings eR and $(1-e)R$ such that $R\text{-mod}$ is equivalent to $eR\text{-mod} \times (1-e)R\text{-mod}$. Furthermore, for any R -module M , there is a natural isomorphism $M \cong eM \oplus (1-e)M$. A central idempotent is precisely the same data as an R - R -bimodule map from R to itself. Hence, the proposition below is the ring spectrum version of this algebraic result.

Proposition 7.2 *Let $R \in \mathcal{S}$ be a ring spectrum whose underlying spectrum is cofibrant and let e be an idempotent of $[R, R]^{(R, R)}$. Then there is a Quillen equivalence*

$$\Delta : R\text{-mod} \rightleftarrows L_{\Gamma e R} R\text{-mod} \times L_{\Gamma(1-e)R} R\text{-mod} : \prod.$$

Proof We construct eR in the category of R - R -bimodules and then consider it as a right R -module. Since R is cofibrant, it follows that eR is cofibrant as a right R -module (see below for details). We localise the category of R -modules at the cell right R -module $\Gamma e R$ and note that the weak equivalences of $L_{\Gamma e R} R\text{-mod}$ are the $(eR)^R$ -equivalences. We can then follow the proof of Theorem 4.4. ■

There is a forgetful functor U from R - R -bimodules to $R\text{-mod}$, this is a right Quillen functor with left adjoint $M \mapsto M \wedge R$. Take $f : A \rightarrow B$ a generating (acyclic) cofibration of \mathcal{S} . Then $g = \text{Id}_R \wedge f \wedge \text{Id}_R$ is a generating (acyclic) cofibration for the category of R - R -bimodules. Since $f \wedge \text{Id}_R$ is a cofibration of spectra, it follows that g is a cofibration of left R -modules, hence U is a left Quillen functor. A slight alteration of this argument shows that a cofibrant R - R -bimodule is cofibrant as a right R -module.

The functor U induces a ring map $[R, R]^{(R, R)} \rightarrow [R, R]^R \cong \pi_0(R)$. If R is commutative, every R -module can be considered as an R - R -bimodule, this defines a right Quillen

functor I . Let M be an R - R -bimodule with actions ν and ν' . Then define SM as the coequaliser: $R \wedge M \xrightleftharpoons[\nu' \circ \tau]{\nu} M \longrightarrow SM$. It follows that S is the left adjoint of I and that UI is the identity functor of R -mod.

These functors give a retraction: $[R, R]^R \xrightarrow{I} [R, R]^{(R, R)} \xrightarrow{U} [R, R]^R$. Thus in the commutative case it is no restriction to consider an idempotent $e \in [R, R]^{(R, R)}$. The Quillen equivalence above would then follow from our main result and would be a strong symmetric monoidal Quillen equivalence.

For E a cofibrant spectrum and R a commutative ring spectrum, the $L_{E \wedge R}$ -model structure on the category of R -modules has weak equivalences those maps f which are E -equivalences of underlying spectra. Thus $L_{E \wedge R} R$ -mod is precisely the model category of R -modules in $L_E \mathcal{S}$.

One important source of idempotents in $\pi_0(R)$ (or $[R, R]^{(R, R)}$) is the image of idempotents in $\pi_0(S)$ via the unit map $S \rightarrow R$. We return to our primary example of rational equivariant EKMM-spectra to give an example of this. To obtain our commutative ring spectrum we use [EKMM97, Chapter VIII, Theorem 2.2], we give the statement that we will need below. Here we assume that E is a cell spectrum (hence cofibrant).

Theorem 7.3 *For a cell commutative R -algebra A , the localisation $\lambda : A \rightarrow A_E$ can be constructed as the inclusion of a subcomplex in a cell commutative R -algebra A_E . In particular $A \rightarrow A_E$ is an E -equivalence and a cofibration of commutative ring spectra for any cell commutative R -algebra A .*

Definition 7.4 *Let $S_{\mathbb{Q}}$ be the commutative ring spectrum constructed as the $S_{\mathcal{M}}^0 \mathbb{Q}$ -localisation of S .*

It follows immediately that the unit $\eta : S \rightarrow S_{\mathbb{Q}}$ is an $S_{\mathcal{M}}^0 \mathbb{Q}$ -equivalence. Thus, by our universal property for $S_{\mathcal{M}}^0 \mathbb{Q}$ (Lemma 5.9) and the fact that $S_{\mathbb{Q}}$ has rational homotopy groups, we have the first statement of the following result. The rest of the lemma follows by a standard argument, see [Ada74, 13.1].

Lemma 7.5 *There is a weak equivalence $S_{\mathcal{M}}^0 \mathbb{Q} \rightarrow S_{\mathbb{Q}}$. Hence all $S_{\mathbb{Q}}$ -modules are $S_{\mathcal{M}}^0 \mathbb{Q}$ -local and so all $S_{\mathbb{Q}}$ -modules have rational homotopy groups.*

Theorem 7.6 *There is a strong symmetric monoidal Quillen equivalence:*

$$S_{\mathbb{Q}} \wedge (-) : G\mathcal{M}_{\mathbb{Q}} \rightleftarrows S_{\mathbb{Q}}\text{-mod} : U.$$

Proof The above functors form a strong monoidal Quillen pair (with the usual structure on $G\mathcal{M}$). Since cofibrations are unaffected by localisation, $S_{\mathbb{Q}} \wedge (-) : G\mathcal{M}_{\mathbb{Q}} \rightarrow S_{\mathbb{Q}}\text{-mod}$ preserves cofibrations. Consider an acyclic rational cofibration $X \rightarrow Y$, we know that $S_{\mathbb{Q}} \wedge (-)$ applied to this gives a cofibration, we must check that it is also a π_* -isomorphism.

We see that $X \wedge S_{\mathcal{M}}^0 \mathbb{Q} \rightarrow Y \wedge S_{\mathcal{M}}^0 \mathbb{Q}$ is a cofibration and a π_* -isomorphism, so in turn $X \wedge S_{\mathcal{M}}^0 \mathbb{Q} \wedge S_{\mathbb{Q}} \rightarrow Y \wedge S_{\mathcal{M}}^0 \mathbb{Q} \wedge S_{\mathbb{Q}}$ is a π_* -isomorphism (by the monoid axiom). This proves that $X \wedge S_{\mathbb{Q}} \rightarrow Y \wedge S_{\mathbb{Q}}$ is a $\pi_*^{\mathbb{Q}}$ -isomorphism between $S_{\mathbb{Q}}$ -modules, which we

know have rational homotopy groups and thus this map is in fact a π_* -isomorphism. Hence we have a Quillen pair, now we prove that it is a Quillen equivalence. The right adjoint preserves and detects all weak equivalences. The map $X \rightarrow S_{\mathbb{Q}} \wedge X$ is a rational equivalence for all cofibrant S -modules X . This follows since smashing with a cofibrant object will preserve the $\pi_*^{\mathbb{Q}}$ -isomorphism $S \rightarrow S_{\mathbb{Q}}$. ■

It follows that we have an isomorphism of rings $[S, S]_{\mathbb{Q}}^G \rightarrow [S_{\mathbb{Q}}, S_{\mathbb{Q}}]^{S_{\mathbb{Q}}\text{-mod}}$. Hence for an idempotent e of the rational Burnside ring we can split $S_{\mathbb{Q}}\text{-mod}$ using the objects $eS \wedge S_{\mathbb{Q}}$ and $(1 - e)S \wedge S_{\mathbb{Q}}$. We can then apply Theorem 3.5 to see that the strong symmetric monoidal adjunction below is a Quillen equivalence, hence we have a comparison between our splitting of $S_{\mathbb{Q}}\text{-mod}$ and Corollary 6.3.

$$S_{\mathbb{Q}} \wedge (-) : L_{eS}GM_{\mathbb{Q}} \xleftrightarrow{\quad} L_{(\Gamma eS) \wedge S_{\mathbb{Q}}} S_{\mathbb{Q}}\text{-mod} : U$$

We briefly wish to mention that following the construction of $S_{\mathbb{Q}}$ one can make R_e for any commutative ring R and idempotent $e \in \pi_0(R)$ by localising R at ΓeR . It follows that R_e is weakly equivalent to ΓeR and hence any R_e -module is ΓeR -local. Then, as with the $S_{\mathbb{Q}}$ -case, one can prove that extension and restriction of scalars along $R \rightarrow R_e$ induces a Quillen equivalence between $L_{\Gamma eR}R\text{-mod}$ and $R_e\text{-mod}$. This is a manifestation of [Wol98, Theorem 2]. Hence we have a different statement of the splitting result: there is a Quillen equivalence $R\text{-mod} \xleftrightarrow{\quad} R_e\text{-mod} \times R_{1-e}\text{-mod}$, induced by extension and restriction of scalars.

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